Indian Statistical Institute, Bangalore Centre

M.Math II Year, Second Semester

Semestral Examination

Advanced Probability

May 6, 2013 Time: 2.00 PM - 5.00 PM Instructor: B. Rajeev

Maximum Marks : 100

1. Let $\{\mu_t, t > 0\}$ be a family of probability measures on $(\mathbb{R}, \mathcal{B})$. Show that $\{\mu_t, t > 0\}$ is a convolution semi-group iff there exists a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a stochastic process $(X_t)_{t \geq 0}$ on it such that (X_t) has stationary and independent increments and $\mu_t(A) = P(X_t \in A)$. [12]

2. Construct a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and random variables $X, Y_1, Y_2, \cdots Y_n, \cdots$ on it such that X is uniformly distributed on $[0, 1]$ and given $X = p$, $Y_1, \cdots Y_n$ are i.i.d. Ber.(p) random variables for all $n \geq 1$. [12]

3. Suppose $0 < p < 1$ and $\{X_i\}_{i \geq 1}$ are i·i·d· random variables with $P(X_i = 1) = p = 1 - P\{X_i = -1\}$. Let $S_n := \sum\limits_{i=1}^{n} X_i$

   (a) Show that if $p = \frac{1}{2}, P\{\overline{\lim\limits_{n\to\infty}} S_n = \infty\} = P\{\underline{\lim\limits_{n\to\infty}} S_n = -\infty\} = 1$.

   (b) Calculate the $P\{\overline{\lim\limits_{n\to\infty}} S_n = \infty\}$ when $p \neq \frac{1}{2}$.

   [8+7]

4. Let X be a square integrable martingale with square variation $\langle X \rangle$. Let τ be a finite stopping time. Show the following :

   (a) If $E\langle X \rangle_\tau < \infty$ then
   $E(X_\tau - X_0)^2 = E\langle X \rangle_\tau$ and $EX_\tau = EX_0$.

   (b) If $E\langle X \rangle_\tau = \infty$ then both equalities in (a) may fail.

   [6+7]

5. Let $\{Y_i\}_{i \geq 1}$ be an i.i.d. sequence with $EY_i = 0$ and $\text{Var}(Y_i) = 1, i = 1, 2, \cdots$. Let $\{Z_i\}$ be a sequence of independent random variables with

$$P(Z_i = i) = P(Z_i = -i) = \frac{1}{2}(1 - P(Z_i = 0)) = \frac{1}{2i^2}$$

   $i = 1, 2, \cdots$. Let $X_i := Y_i + Z_i$ and $S_n := X_1 + \cdots + X_n$

   (a) Show that $\frac{S_n}{\sqrt{n}}$ converges in distribution to the standard normal distribution.

   (b) Give an example of the sequence $\{Y_i\}$ to show that the Lindberg condition fails for $\{X_i\}$.

   [10+10]

6. (a) Let (X, Y) have joint density $f(x, y)$. Let $h : \mathbb{R} \to \mathbb{R}$ measurable with $E \mid h(X)| < \infty$. Show that, almost surely,

$$E[h(X) \mid Y] = \frac{\int h(x) f(x, Y) dx}{\int f(x, Y) dx}$$

(b) Show that the conditional Markov inequality is true : If $f : [0, \infty) \to [0, \infty)$ is non decreasing and $\epsilon > 0$ then

$$P(|X| > \epsilon | \mathcal{F}) \leq \frac{E[f(|X|) \mid \mathcal{F}]}{f(\epsilon)}$$

for random variables X with $Ef(\mid X \mid) < \infty$ and sub σ-fields $\mathcal{F}$.

(c) Let $A \subset \mathbb{R}^n$ have positive Lebesgue measure. Let X be an A-valued random vector having the uniform distribution on A. Let $B \subset A$ have positive Lebesgue measure. Write down explicitly the conditional distribution of X given $\mathcal{F} := \{\phi, \Omega, \{X \in B\}, \{X \in B^c\}\}$.

[10+8+5]

7. Let $\{X_i\}$ be an i· i· d· sequence of non negative random variables. Show that $EX_1 < \infty$ iff almost surely $\lim_{n \to \infty} \frac{X_n}{n} = 0$. [15]

(Hint : Use the Borel-Cantelli lemma).